ALMOST-SURE UNIQUENESS FOR THE SIGNATURE OF SLE CURVES

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Abstract. We prove that, for $0 < \kappa \leq 4$, the signatures of SLE_κ curves determine the curves almost surely.

1. Introduction

The signature of a path is a formal series of its iterated integrals. In [Chen54], K.T. Chen observed that the map that sends a path to its signature forms a homomorphism from the concatenation algebra to the tensor algebra and used it to study the cohomology of loop spaces. Recent interest in the study of signature has been sparked by its role in the rough path theory. In particular, it was shown by Hambly and Lyons in [LH02] that for ODEs driven by paths with bounded total variations, the signature is a fundamental representation of the effect of the driving signal on the solution.

If we consider the signatures of paths as a representation of paths, then an interesting question is whether this representation is faithful. This was first considered by Chen himself [Chen58], who proved that irreducible, piecewise regular continuous paths have the same signature if and only if they are equal up to a translation and a reparametrisation.

His result was generalised with a new, quantitative approach by Hambly and Lyons in [LH02] who showed that two paths γ and $\tilde{\gamma}$ with finite total variations have the same signature if and only if γ can be expressed as the concatenation of $\tilde{\gamma}$ with a tree-like path σ. A continuous function $\sigma : [0,1] \to \mathbb{R}^d$ is tree-like if there exists a continuous function $h : [0,1] \to [0,\infty)$ such that $h(0) = h(1) = 0$ and

$$|\sigma_t - \sigma_s| \leq h(t) + h(s) - 2 \inf_{s \leq u \leq t} h(u).$$

In this article we establishes uniqueness amongst paths γ satisfying the following conditions:

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$.

(1) γ can be reparametrised as a continuous curve $\tilde{\gamma} : [0,1] \to \overline{\mathbb{D}}$, such that $\tilde{\gamma}(0) = -1$, $\tilde{\gamma}(1) = 1$ and $\tilde{\gamma}(0,1) \subset \mathbb{D}$.
(2) γ has finite p-variation for some $1 \leq p < 2$.
(3) γ is a simple curve.

Let $\mathcal{C}_2(-1,1,\mathbb{D})$ denote the set of paths γ satisfying these three conditions. Let $S(\gamma)$ denote the signature (see Definition 4) of a path γ.

Theorem 1. *Let γ, $\gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$. Then $S(\gamma)_{0,1} = S(\gamma')_{0,1}$ if and only if γ and γ' are equal up to a reparametrisation.*

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An interesting, but difficult extension is to prove that if the signatures of two curves with finite $p > 1$ -variations are equal, then the paths are equal up to the tree-like path equivalence. The restriction $1 \leq p < 2$ gives us the existence of signature for free, thanks to Young's integration theory.

Theorem 1 only applies to paths with finite p-variations, where $p < 2$. In particular, our results can only be applied to study stochastic processes whose sample paths are almost surely smoother than the Brownian motion sample paths. One example of such processes is the Chordal SLE$_\kappa$ measure. The SLE measures were born from the study of lattice models which have conformally invariant scaling limit. There are a number of other lattice models whose scaling limit have been proved to be an SLE curve under some boundary conditions, such as the loop erased random walk ($\kappa = 2$, [LSW04]), the Ising model ($\kappa = 3$, [CS12]), the level lines of Gaussian Free Field ($\kappa = 4$, [SS09]), percolation on the triangular lattice ($\kappa = 6$, [CN07] and [Snv01]), and the Peano curve of the uniform spanning tree ($\kappa = 8$, [LSW04]).

The path regularity and, in particular, the roughness of SLE curves, in relation to the speed κ of the driving Brownian motion, is an extremely interesting topic. It is intuitively clear that the SLE curves becomes rougher as the speed of the driving Brownian motion increases. In [JL10], the optimal Hölder exponent of SLE curves under the capacity parametrisation was proved to be

$$\min\left(\frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}\right).$$

In [Bef08], V. Beffara proved that the Hausdorff dimension of SLE curves is $\min\left(1 + \frac{\kappa}{8}, 2\right)$. Therefore, the optimal Hölder exponent cannot exceed $1 + \frac{\kappa}{8}$. B. Werness[Wer12] proved that for $0 < \kappa \leq 4$, almost surely, there exists a parametrisation under which the SLE curve is α-Hölder for any $\alpha < \frac{1}{1+\frac{\kappa}{8}}$. In another words, the roughness of an SLE curve grows linearly with the speed of the driving Brownian motion. It is strongly believed that this remains true for $4 < \kappa < 8$. However, to the best of our knowledge, this problem remains open.

Werness's result allowed him to define the signatures of SLE curves using Young's integral and used Green's theorem to compute the first three gradings of the expected signature of SLE curves.

In [LQ12], Le Jan and Qian proved that the signatures of Brownian motion sample paths determine the Brownian motion paths almost surely. The result was extended to diffusion processes in [Geng13]. Both results rely on the Strong Markov property. Although the Chordal SLE$_\kappa$ measure is not Markov, the inversion problem can be tackled for $\kappa \leq 4$ since the Chordal SLE$_\kappa$ measure is supported on simple curves from -1 and 1. Therefore, it follows from Theorem 1 from the last chapter that:

Theorem 2. *Let $0 < \kappa \leq 4$. Let $\mathbb{P}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that if $\gamma, \gamma' \in \mathcal{N}^c$ and $S(\gamma)_{0,1} = S(\gamma')_{0,1}$, then γ and γ' are equal up to a reparametrisation.*

The plan for the rest of the article is as follows.

In section 2, we recall the definition and properties of signature.

In section 3, we prove Theorem 1.

In section 4, we prove Theorem 2.

2. The Signature of a path

2.1. **Basic notations.** Let $T^n\left(\mathbb{R}^d\right)$ and $T\left(\mathbb{R}^d\right)$ denote the graded algebras on $\mathbb{R}^d$ defined by

$$T^n\left(\mathbb{R}^d\right) := \oplus_{k=0}^{n}\left(\mathbb{R}^d\right)^{\otimes k}$$

and

$$T\left(\mathbb{R}^d\right) := \oplus_{k=0}^{\infty}\left(\mathbb{R}^d\right)^{\otimes k}$$

where $\left(\mathbb{R}^d\right)^{\otimes 0} := \mathbb{R}$. We equip $T\left(\mathbb{R}^d\right)$ with the Euclidean metric by identifying $\left(\mathbb{R}^d\right)^{\otimes k}$ with $\mathbb{R}^{d^k}$.

Let $p > 1$ and let $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ denote the set of all continuous functions $\gamma : [0,T] \to \mathbb{R}^d$ such that

$$\|\gamma\|^p_{\mathcal{V}^p([0,T],\mathbb{R}^d)} := \sup_{\mathcal{P}} \sum_k \left|\gamma_{t_{k+1}} - \gamma_{t_k}\right|^p < \infty. \tag{2.1}$$

where the supremum is taken over all finite partitions $\mathcal{P} := (t_0, t_1, .., t_{n-1}, t_n)$, where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$.

The elements of $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ will be called curves with finite p-variation.

Note that $\|\cdot\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}$ defines a norm on $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$.

2.2. **Young's integrals, signature and approximation theorems.** While the Lebesgue-Stieltjes integration theory is sufficient to define integrals against functions with finite 1-variations, we need Young's integration theory to define integrals against functions with finite p variations, where $1 \le p < 2$. A proof of the following criterion for the existence of Young's integral can be found in [LCL04]:

Theorem 3. *(L. Young*[Yng36]*)Let $p, q \ge 1$ be such that $\frac{1}{p} + \frac{1}{q} > 1$, $T > 0$ and $t \in [0,T]$. Then for $\gamma_\cdot \in \mathcal{V}^p([0,t],\mathbb{R})$ and $\gamma'_\cdot \in \mathcal{V}^q([0,t],\mathbb{R})$, the following limit exists:*

$$\int_0^t \gamma \mathrm{d}\gamma' := \lim_{\|\mathcal{P}\|\to 0} \sum \gamma_{t_i}\left(\gamma'_{t_{i+1}} - \gamma'_{t_i}\right) \tag{2.2}$$

where the sum is over partition points $0 = t_0 < t_1 < .. < t_n = t$ in $\mathcal{P}$, and $\|\mathcal{P}\| := \sup_i |t_{i+1} - t_i|$.

Furthermore, as a function of t, $t \to \int_0^t \gamma \mathrm{d}\gamma' \in \mathcal{V}^q([0,T],\mathbb{R})$, and there is a constant $C_{p,q} > 0$ depending only on p and q such that

$$\left\|\int_0^{\cdot} (\gamma_s - \gamma_0)\,\mathrm{d}\gamma'_s\right\|_{\mathcal{V}^q([0,T],\mathbb{R})} \le C_{p,q}\,\|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R})}\,\|\gamma'\|_{\mathcal{V}^q([0,T],\mathbb{R})}\,. \tag{2.3}$$

The limit in (2.2) is called the *Young's integral of γ with respect to γ'*. In the case when γ' has finite 1-variation, the integral coincides with the Lebesgue-Stieltjes integral.

Theorem 3 allows us to define the signature of $\gamma \in \mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ for $1 \le p < 2$. We will use $\triangle_n(s,t)$ to denote the set $\{(t_1, .., t_n) : s < t_1 < ... < t_n < t\}$ and $\triangle$ to denote the set $\{(s,t) : 0 \le s \le t \le T\}$.

Definition 4. Let $1 \le p < 2$ and $\gamma \in \mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$. The *lift* of γ is a function $S(\cdot) : \triangle \to T\left(\mathbb{R}^d\right)$, defined by

$$S(\gamma)_{s,t} := 1 + \sum_{n=1}^{\infty} \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1} \otimes ... \otimes \mathrm{d}\gamma_{t_n} \tag{2.4}$$

where the sum $+$ is the direct sum operation in $T\left(\mathbb{R}^d\right)$.

The signature of γ is defined as $S\left(\gamma\right)_{0,T}$.

Remark 5. $S\left(\gamma\right)_{\cdot,\cdot}$ is well-defined for $\gamma \in \mathcal{V}^p\left(\left[0,T\right],\mathbb{R}^d\right)$, with $p<2$ because, for instance, $\int \mathrm{d}\gamma \otimes \mathrm{d}\gamma$ exists because $\frac{1}{p}+\frac{1}{p}>1$. Moreover, the resulting integral remains in $\mathcal{V}^p\left(\left[0,T\right],\mathbb{R}^d\right)$, so the resulting integral can be integrated again with respect to γ.

Remark 6. Let $\mathbf{e}_1,..,\mathbf{e}_d$ be the standard basis in $\mathbb{R}^d$. If $\gamma_\cdot = \sum_{i=1}^d \gamma_\cdot^{(i)}\mathbf{e}_i$, then (2.4) can also be written as

$$S\left(\gamma\right)_{s,t} := 1+\sum_{n=1}^{\infty}\sum_{1\leq i_1,..,i_n\leq d}\int_{\triangle_n(s,t)}\mathrm{d}\gamma_{t_1}^{(i_1)}...\mathrm{d}\gamma_{t_n}^{(i_n)}\mathbf{e}_{i_1}\otimes...\otimes\mathbf{e}_{i_n}$$

The term $\int_{\triangle_n(s,t)}\mathrm{d}\gamma_{t_1}^{(i_1)}...\mathrm{d}\gamma_{t_n}^{(i_n)}$ will be denoted by $\gamma^{i_1 i_2..i_n}$ if no confusion is possible.

Remark 7. The signatures of rougher paths, if exist, will have to be defined using the Rough Path theory. We shall not need it in this chapter.

We shall use the following properties of signature, whose proofs can be found in [LCL04].

1. (Invariance under reparametrisation)For any $t\in[0,\infty)$, $S\left(\gamma\right)_{0,t}$ is invariant under any reparametrisation of γ on $[0,t]$.

2. (Inverse) $S\left(\gamma\right)_{0,T}\otimes S\left(\overleftarrow{\gamma}\right)_{0,T}=\mathbf{1}$, where $\overleftarrow{\gamma}\left(t\right):=\gamma\left(T-t\right)$ is the reversal of γ and $\mathbf{1}$ is the identity element in $T\left(\mathbb{R}^d\right)$.

3. (Chen's Identity)$S\left(\gamma\right)_{s,u}\otimes S\left(\gamma\right)_{u,t}=S\left(\gamma\right)_{0,t}$ for any $0\leq s<u<t\leq T$

4. (Scaling and translation)Let $\lambda\in\mathbb{R}^d$, $\mu\in\mathbb{R}$, then

$$S\left(\lambda+\mu\gamma\right)_{s,t}=1+\sum_{n=1}^{\infty}\mu^n\int_{\triangle_n(s,t)}\mathrm{d}\gamma\left(t_1\right)\otimes...\otimes\mathrm{d}\gamma\left(t_n\right)$$

5.(Shuffle product formula) If we define a *shuffle* of $1,...,r$ and $r+1,...,r+s$ to be a permutation of $\{1,2,...,r+s\}$ such that $\sigma\left(1\right)<\sigma\left(2\right)<..<\sigma\left(r\right)$ and $\sigma\left(r+1\right)<...<\sigma\left(r+s\right)$.

Proposition 8. *([LCL04], Theorem 2.15)Let $1\leq p<2$ and $\gamma\in\mathcal{V}^p\left(\left[0,T\right],\mathbb{R}^d\right)$, then using the notation of Remark 6,*

$$\gamma^{k_1...k_r}\cdot\gamma^{k_{r+1}...k_{r+s}}=\sum_{shuffles\ \sigma}\gamma^{k_{\sigma^{-1}(1)}...k_{\sigma^{-1}(r+s)}}$$

where $\cdot$ is the multiplication operation in $\mathbb{R}$.

The sum $\sum_{shuffles\ \sigma}\gamma^{k_{\sigma^{-1}(1)}...k_{\sigma^{-1}(r+s)}}$ is denoted by $\gamma^{k_1...k_r\sqcup k_{r+1}...k_{r+s}}$.

The signature of a curve γ is normally evaluated by first calculating the signature of its piecewise smooth approximation, and then taking limit in $\left\|\cdot\right\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}$. A natural candidate for such an approximation is the piecewise linear approximation. For a continuous function γ and a partition $\mathcal{P}:=t_0=0<t_1<..<t_n=T$, the piecewise linear interpolation of γ with respect to $\mathcal{P}$ is defined as the following function on $[0,T]$:

$$\gamma_t^{\mathcal{P}}:=\gamma_{t_i}+\left(\frac{\gamma_{t_{i+1}}-\gamma_{t_i}}{t_{i+1}-t_i}\right)\left(t-t_i\right)\ \text{for}\ t\in\left[t_i,t_{i+1}\right]$$

Then the following approximation theorem holds:

Lemma 9. *(Lemma 1.12 and Proposition 1.14,* [LCL04]*)Let p and q be such that $1 \leq p < q$. Let $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$. Then for all finite partitions $\mathcal{P}$,*

$$\left\|\gamma^{\mathcal{P}}\right\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)} \leq \|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}$$

Furthermore for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all partitions $\mathcal{P}$ of $[0,T]$ satisfying $\|\mathcal{P}\| < \delta$ we have

$$\begin{aligned} \left\|\gamma - \gamma^{\mathcal{P}}\right\|_{\mathcal{V}^q([0,T],\mathbb{R}^d)} &< \varepsilon, \text{ and} \\ \sup_{t\in[0,T]} \left\|\gamma_t - \gamma_t^{\mathcal{P}}\right\| &< \varepsilon. \end{aligned}$$

We shall need to use Green's theorem to evaluate the signature. In order to ensure that the domain we are integrating over is Jordan, we will approximate our curves by simple curves. This is made possible by the following lemma:

Lemma 10. *(*[Wer12]*, Lemma 4.3) Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous simple curve. Then for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[0,1]$ such that $\|\mathcal{P}\| < \varepsilon$ and $\gamma^{\mathcal{P}}$ is simple.*

The lemma relies crucially on the uniform continuity of γ. The following corollary follows immediately.

Corollary 11. *Let $\gamma : [0,1] \to \overline{\mathbb{D}}$ be a continuous simple curve such that $\gamma_0 = -1$, $\gamma_1 = 1$ and $\gamma_t \subset \mathbb{D}$ for all $t \in (0,1)$. Then for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[0,1]$ such that $\|\mathcal{P}\| < \varepsilon$ and $\gamma^{\mathcal{P}}$ is simple. Furthermore, $\gamma_t^{\mathcal{P}} \subset \mathbb{D}$ for $t \in (0,1)$ and $\gamma_0^{\mathcal{P}} = -1$, $\gamma_1^{\mathcal{P}} = 1$.*

Proof. The only thing to prove is $\gamma^{\mathcal{P}}(0,1) \subset \mathbb{D}$. Since $\gamma_{t_i}, \gamma_{t_{i+1}} \subset \mathbb{D}$ for $i \neq 0$ or $n-1$, thus by the convexity of $\mathbb{D}$, the line segment between γ_{t_i} and $\gamma_{t_{i+1}}$ lies in $\mathbb{D}$. Note also that the line segment between γ_0 and γ_1 and the segment between γ_{n-1} and γ_n also lies in $\mathbb{D}$ by convexity. Thus we have the result. □

The following lemma is extremely useful in proving the properties of Young's integral.

Lemma 12. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous curve with finite p-variation, where $p < 2$. Let $\mathcal{P}_m$ be a sequence of partitions such that $\mathcal{P}_m$ contains both 0 and 1 for all m and $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$. For any $I \in \{1,\ldots,d\}^n$,*

$$\pi^I\left[S(\gamma)_{0,1}\right] = \lim_{m\to\infty} \pi^I\left[S\left(\gamma_s^{\mathcal{P}_m}\right)_{0,1}\right]. \tag{2.5}$$

Proof. See Corollary 2.11 in [LCL04]. □

3. Uniqueness of signature

Recall that the term $\eta^{11\ldots122\ldots2}$ (see Remark 6) with n $1s$ and k $2s$ is an important term in the signature of η for our purpose and will be given the special notation $\eta_{n,k}$. These particular iterated integrals can be reduced to just one line integral, which can be evaluated using Green's theorem. This idea is the content of the following lemma.

Lemma 13. *Let $\eta:[0,1]\to\mathbb{R}^2$ be a positively oriented, simple closed curve with bounded total variation and D be its interior, then*

$$\eta_{n,k}=\frac{1}{(n-1)!}\frac{(-1)^{k-1}}{(k-1)!}\int_D\left(x-\eta_0^{(1)}\right)^{n-1}\left(\eta_0^{(2)}-y\right)^{k-1}\mathrm{d}x\mathrm{d}y \tag{3.1}$$

Proof. Let $\eta^{(1)}$and $\eta^{(2)}$ be the first and second coordinate components of η respectively.

Recall that

$$\eta_{n,k}=\int_0^1\mathrm{d}\eta_{s_1}^{(1)}\ldots\mathrm{d}\eta_{s_n}^{(1)}\mathrm{d}\eta_{s_{n+1}}^{(2)}\ldots\mathrm{d}\eta_{s_{n+k}}^{(2)}.$$

The key idea here is to integrate with respect to $\eta^{(1)}$s first and then integrate the $\eta^{(2)}$s.

$$\begin{aligned}
\eta_{n,k} &= \int\ldots\int_{0<t_1<..<t_n<s_1<\ldots<s_k<1}\mathrm{d}\eta_{t_1}^{(1)}\ldots\mathrm{d}\eta_{t_n}^{(1)}\mathrm{d}\eta_{s_1}^{(2)}\ldots\mathrm{d}\eta_{s_k}^{(2)}\\
&= \int_{0<s_1<\ldots<s_k<1}\frac{1}{n!}\left(\eta_{s_1}^{(1)}-\eta_0^{(1)}\right)^n\mathrm{d}\eta_{s_1}^{(2)}\ldots\mathrm{d}\eta_{s_k}^{(2)}\\
&= \int_0^1\int_{s_1}^1\ldots\int_{s_{k-2}}^1\int_{s_{k-1}}^1\frac{1}{n!}\left(\eta_{s_1}^{(1)}-\eta_0^{(1)}\right)^n\mathrm{d}\eta_{s_k}^{(2)}\ldots\mathrm{d}\eta_{s_1}^{(2)}\text{ by Fubini's theorem}\\
&= \frac{1}{n!}\frac{1}{(k-1)!}\int_0^1\left(\eta_{s_1}^{(1)}-\eta_0^{(1)}\right)^n\left(\eta_1^{(2)}-\eta_{s_1}^{(2)}\right)^{k-1}\mathrm{d}\eta_{s_1}^{(2)}\\
&= \frac{1}{(n-1)!}\frac{1}{(k-1)!}\int_D\left(x-\eta_0^{(1)}\right)^{n-1}\left(\eta_1^{(2)}-y\right)^{k-1}\mathrm{d}x\mathrm{d}y\text{ by Green's Theorem}
\end{aligned}$$

where D is the interior of η. □

Remark 14. B. Werness [Wer12] is the first to realise that the Green's theorem can be used to compute some terms in the signature of a curve. He used it to compute the first three gradings of the expected signature of SLE curve.

Note that Lemma 13 cannot be applied directly to elements of $\mathcal{C}_2(-1,1,\mathbb{D})$ as they are not closed curves and are "too rough". For the first problem, we need to "closed off" the curves in $\mathcal{C}_2(-1,1,\mathbb{D})$ by concatenating it with the upper semi-circular boundary of $\mathbb{D}$ from 1 to -1. Let us describe this more precisely.

Recall that $\mathcal{C}_2(-1,1,\mathbb{D})$ is defined just before Theorem 1.

Let ϕ denote the anti-clockwise semi-circular boundary of $\mathbb{D}$, or more precisely, $\phi(t):=(\cos t,\sin t),0\leq t\leq\pi$.

Let $p\geq 1$. For elements γ and $\tilde{\gamma}$ in $\mathcal{V}^p\left([0,T_2],\mathbb{R}^d\right)$ and $\mathcal{V}^p\left([0,T_1],\mathbb{R}^d\right)$, define a concatenation product $\star$:$\mathcal{V}^p\left([0,T_2],\mathbb{R}^d\right)\times\mathcal{V}^p\left([0,T_1],\mathbb{R}^d\right)\to\mathcal{V}^p\left([0,T_1+T_2],\mathbb{R}^d\right)$ by

$$\begin{aligned}
\gamma\star\tilde{\gamma}(u) &:= \gamma(u), \quad u\in[0,T_1],\\
\gamma\star\tilde{\gamma}(u) &:= \tilde{\gamma}(u-T_1)+\gamma(T_1)-\tilde{\gamma}(0), \quad u\in[T_1,T_1+T_2]
\end{aligned}$$

Then for $\gamma\in\mathcal{C}_2(-1,1,\mathbb{D})$, $\eta=\gamma\star\phi$ is a simple closed curve.

As η does not in general has bounded total variation, we will prove a version of Lemma 13 that works for η.

Lemma 15. *Let $\gamma\in\mathcal{C}_2(-1,1,\mathbb{D})$. If $\eta=\gamma\star\phi$, then (3.1) holds for η.*

Proof. Let $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$. By Lemma 11, there exists a sequence of partitions $(\mathcal{P}_m)_{m=1}^{\infty}$ such that $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$, $\gamma_0^{\mathcal{P}_m} = -1$, $\gamma_1^{\mathcal{P}_m} = 1$ and $\gamma^{\mathcal{P}_m} \star \phi$ is a simple closed curve. Note in particular that $\gamma^{\mathcal{P}_m} \star \phi$ is a simple closed curve and let D_m be the interior of $\gamma^{\mathcal{P}_m} \star \phi$, which we shall denote as $\eta^{(m)}$.

Since $\gamma^{\mathcal{P}_m} \to \gamma$ in $\|\cdot\|_{\infty}$, we have for each $(x,y) \in \mathbb{R}^2 \backslash \eta \cup_{m=1}^{\infty} \eta^{\mathcal{P}_m}$, $1_{D_m}(x,y) \to 1_D(x,y)$, where D is the interior of η. As γ has finite p-variation, where $p < 2$, γ can be reparametrised to be a $\frac{1}{p}$-Hölder continuous path ([LCL04], Section 1.2.2) and hence $\gamma[0,1]$ has Hausdorff dimension strictly less than 2. Therefore, the set $\eta \cup_{m=1}^{\infty} \eta^{\mathcal{P}_m}$ has two dimensional Lebesgue measure is zero. Thus $1_{D_m} \to 1_D$ almost everywhere on $\mathbb{R}^2$ in Lebesgue measure.

By the bounded convergence theorem,

$$\begin{aligned} & \lim_{m\to\infty} \int_{\mathbb{R}^2} (x+1)^{n-1} (-y)^{k-1} 1_{D_m} \mathrm{d}x\mathrm{d}y \\ = & \int_{\mathbb{R}^2} (x+1)^{n-1} (-y)^{k-1} 1_D \mathrm{d}x\mathrm{d}y \end{aligned}$$

By Lemma 12,

$$\lim_{m\to\infty} \gamma_{n,k}^{\mathcal{P}_m} = \gamma_{n,k}.$$

(3.1) then follows from Lemma 13. ◻

Before proving our main result, we need just one more technical lemma.

Lemma 16. *Let $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$. If $\gamma[0,1] = \gamma'[0,1]$, then there exists a continuous strictly increasing function $r(t)$ such that*

$$\gamma_{r(t)} = \gamma'_t$$

for all $t \in [0,1]$.

Proof. Let γ^{-1} denote the inverse of the function $t \to \gamma_t$, which exist as γ is a simple curve.

Define a function $r : [0,1] \to [0,1]$ by $r(t) = \gamma^{-1} \circ \gamma'(t)$.

As both γ and γ' are injective continuous functions and $\gamma[0,1] = \gamma'[0,1]$, thus r is a bijective continuous function from $[0,1]$ to $[0,1]$. Hence it is monotone.

But $\gamma_0 = \gamma'_0 = -1$, $\gamma_1 = \gamma'_1 = 1$, so $r(0) = 0$ and $r(1) = 1$. Hence r is an increasing function and the result follows. ◻

We now prove the main result of this chapter.

Proof. (of Theorem 1)The only if direction follows from the invariance of signature under reparametrisation.

Let $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$ be such that $S(\gamma) = S(\gamma')$.

Let $\eta := \gamma \star \phi$ and$\eta' := \gamma' \star \phi$. By Chen's identity, $S(\gamma) = S(\gamma')$ implies

$$S(\eta)_{0,1} = S(\eta')_{0,1}.$$

Let D and D' be the interior of η and η' respectively.

Since $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$, we have by Lemma 15 that for $\phi = \eta, \eta'$, $A = D, D'$,

$$\phi_{n,k} = \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \int_A (x+1)^{n-1} y^{k-1} \mathrm{d}x\mathrm{d}y \tag{3.2}$$

Then $S(\eta)_{0,1} = S(\eta')_{0,1}$ implies that

$$\int_{\mathbb{R}^2}(x+1)^{n-1}y^{k-1}1_D(x,y)\,\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^2}(x+1)^{n-1}y^{k-1}1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all n and k.

Thus

$$e^{i\lambda_1}\int_{\mathbb{R}^2}e^{i\lambda_1 x+i\lambda_2 y}1_D(x,y)\,\mathrm{d}x\mathrm{d}y=e^{i\lambda_1}\int_{\mathbb{R}^2}e^{i\lambda_1 x+i\lambda_2 y}1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all $(\lambda_1,\lambda_2)\in\mathbb{R}^2$.

By the fact that Fourier transform is injective on $L^1\left(\mathbb{R}^2\right)$,

$$1_D(x,y)=1_{D'}(x,y)\tag{3.3}$$

for almost all $(x,y)\in\mathbb{R}^2$.

Therefore, both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are null sets with respect to the Lebesgue measure.

As both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are open, so they must both be empty. This means $D\subset\overline{D'}$ and $D'\subset\overline{D}$. Thus $\overline{D}=\overline{D'}$.

Note that as γ,γ' are simple curves and $\gamma,\gamma'\subset\mathbb{D}$ except at the endpoints, the domains D and D' are Jordan domains. Using the Jordan curve theorem, we can prove that $\overline{\mathbb{R}^2\backslash\overline{D}}=\mathbb{R}^2\backslash D$ for any Jordan domain D.

As $\overline{D}=\overline{D'}$, we have $\mathbb{R}^2\backslash D=\mathbb{R}^2\backslash D'$, implying that $\gamma\left[0,1\right]=\gamma'\left[0,1\right]$. The result follows from Lemma 16. □

4. Signature Inversion for Schramm-Loewner Evolution

Let $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P}\right)$ be a filtered probability space. Let $(B_t:t\geq0)$ be a one-dimensional standard Brownian Motion. Let $0<\kappa$. Let $z\in\overline{\mathbb{H}}\backslash\{0\}$. For each $\omega\in\Omega$, consider the initial value problem:

$$\frac{\mathrm{d}g_t(z,\omega)}{\mathrm{d}t}=\frac{2}{g_t(z,\omega)-\sqrt{\kappa}B_t(\omega)}\quad g_0(z)=z\tag{4.1}$$

We shall recall the following facts about g_t from [RS05].

(1) For each ω, a unique solution to this equation exists up to time $T_z>0$, where T_z is the first time such that $g_t-\sqrt{\kappa}B_t\to0$ as $t\to T_z$.

(2) Define

$$H_t=\{z\in\mathbb{H}:t<T_z\}\ \text{ and }K_t=\mathbb{H}\backslash H_t$$

Then H_t is open and simply connected.

(3) For each time $t>0$, g_t defines a conformal map from H_t onto $\mathbb{H}$. In particular, g_t is invertible.

(4) Let $\hat{f}_t(z):=g_t^{-1}(z+\sqrt{\kappa}B_t)$. There exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for all $\omega\in\mathcal{N}^c$, the limit

$$\hat{\gamma}(t,\omega):=\lim_{z\to0,z\in\mathbb{H}}\hat{f}_t(z)$$

exists and $t\to\hat{\gamma}(t)$ is continuous.The two dimensional stochastic process $(\hat{\gamma}_t:t\geq0)$ is called the *Chordal SLE$_\kappa$ curve.*

The Loenwer correspondence from a continuous path $t \to B_t(\omega)$ to $t \to \hat{\gamma}(\cdot, \omega)$ is in fact deterministic and one-to-one. Therefore, the measure on the Brownian paths induces, through this correspondence, a measure on paths in $\overline{\mathbb{H}}$ from 0 to ∞, which we shall call the Chordal SLE$_\kappa$ measure in $\mathbb{H}$.

Proposition 17. *Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$. Then with probability one, the following holds:*

1. ([Wer12], Section 4.1)If $0 < \kappa \leq 4$, then for any $p < 1 + \frac{\kappa}{8}$, γ has finite p-variation.

2.([RS05], Theorem 7.1 and Theorem 6.1)$\gamma : [0, \infty) \to \overline{\mathbb{H}}$ satisfies $\gamma_0 = 0$ and $\liminf_{t \to \infty} \gamma_t = \infty$.

3.([RS05], Theorem 6.1)For $0 \leq \kappa \leq 4$, $t \to \hat{\gamma}_t$ is a simple curve and $\hat{\gamma}(0, \infty) \subset \mathbb{H}$

The fact that $\lim_{t \to \infty} \hat{\gamma}_t = \infty$ *a.s.* means that the signature $S(\hat{\gamma})_{0,\infty}$ will not be defined. Therefore, we shall follow [Wer12] and opt to study the Chordal SLE$_\kappa$ curve in the unit disc $\mathbb{D}$, from -1 to 1. The Chordal SLE_κ measure in domain $\mathbb{D}$ with marked points -1 and 1 is defined as follows:

Definition 18. For $\kappa > 0$. Let $\mathbb{P}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$, D be a simply connected subdomain of $\mathbb{C}$, $a, b \in \partial D$ and f be a conformal map from $\mathbb{H}$ to D, with $f(0) = a$ and $f(\infty) = b$. Then the Chordal SLE$_\kappa$ measure in D with marked points a and b is defined as the measure $\mathbb{P} \circ f^{-1}$.

Remark 19. Although there is a one dimensional family of conformal maps f such that f maps $\mathbb{H}$ to D, 0 to a and ∞ to b, the scale invariance of the Chordal SLE measure in $\mathbb{H}$ means that the measure $\mathbb{P} \circ f^{-1}$ is the same no matter which member f in this one dimensional family we use.

We now prove our almost sure uniqueness theorem concerning the signature of SLE curves.

Proof. (of Theorem 2)Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then by Proposition 17, there exists a $\mathbb{P}_\kappa$-null set $\mathcal{N}$, such that for all $\gamma \in \mathcal{N}^c$,

1. $\gamma(0) = -1$, $\gamma(1) = 1$ and $\gamma(0, 1) \subset \mathbb{D}$.
2. γ has with finite $1 \leq p < 2$ variations.
3. γ is simple.

Therefore, in particular, $\mathcal{N}^c \subset \mathcal{C}_2(-1, 1, \mathbb{D})$.

Let $\gamma, \gamma' \in \mathcal{N}^c$ be such that $S(\gamma) = S(\gamma')$, then by Theorem 1, γ and γ' are reparametrisations of each other. □

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